No. 138/2026, 50–67
ISSN 2657-6988 (online)
ISSN 2657-5841 (printed)
DOI: 10.26408/138.05

Submitted: 16.09.2025
Accepted: 10.02.2026
Published: 30.06.2026

OPTIMAL TIME MOMENTS IN A UNIFORM 1-BULLET SILENT DUEL WITH SCALED EXPONENTIALLY-CONVEX ACCURACY

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Abstract: The finite 1-bullet silent duel is considered, involving two duelists who shoot with exponentially-convex accuracy through a uniformly quantized time. The duel is a symmetric matrix game whose optimal value is 0, and each of the duelists has the same optimal behavior, whether it is in pure or mixed strategies. The actual beginning is never optimal in the duel. Apart from the very end of the duel, the conditions for the optimal time moment existence are found. Numerical experiments confirm that the optimality can be manipulated by changing the accuracy factor that scales the payoffs. The results are applicable in systems under limited or censored communication with uncertainty, latency, and lucrative delayed actions. Some examples of such set-ups are time-sensitive information release (privacy and censorship), queueing and load balancing (information science and telecommunication systems), and block proposal timing for decentralized consensus protocols (in Proof-of-Work and Proof-of-Stake).

Keywords: uniform 1-bullet silent duel, scaled accuracy, exponentially-convex accuracy, matrix game, optimal time moment.

1. UNIFORM 1-BULLET SILENT DUELS

Uniform 1-bullet silent duels are used to model two-sided competitive behavioral patterns, where the purpose of each side (alternatively referred to as the duelist) is to gain a reward by making the best possible decision through quantized time, $T_N = \left\{ t_q \right\}_{q=1}^{N}$ [Aliprantis & Chakrabarti, 2000; Epstein, 2013; Ewerhart, 2020]. The quantized time consists of N successive time moments $\left\{ t_q \right\}_{q=1}^{N}$ of possible shooting [Reinganum, 1989; Romanuke, 2024a; Romanuke, 2025a] and usually represents the standardized time span $[0; 1]$ upon its equidistant (uniform) quantization with a step of $\frac{1}{N-1}$. Number N determines the duel size, where $N \in \mathbb{N} \setminus \{1, 2\}$. Therefore, in a uniform 1-bullet silent duel

$$\langle X_N, Y_N, \mathbf{U}_N \rangle = \left\langle \{x_i\}_{i=1}^{N}, \{y_j\}_{j=1}^{N}, \mathbf{U}_N \right\rangle \tag{1}$$

with duelists' pure strategies sets

$$X_N = \{x_i\}_{i=1}^{N} = \left\{\frac{i-1}{N-1}\right\}_{i=1}^{N} = T_N = \{t_q\}_{q=1}^{N} = \left\{\frac{q-1}{N-1}\right\}_{q=1}^{N} \subset [0; 1] \tag{2}$$

and

$$Y_N = \{y_j\}_{j=1}^{N} = \left\{\frac{j-1}{N-1}\right\}_{j=1}^{N} = T_N = \{t_q\}_{q=1}^{N} = \left\{\frac{q-1}{N-1}\right\}_{q=1}^{N} \subset [0; 1] \tag{3}$$

and payoff matrix

$$\mathbf{U}_N = \left[u_{ij}\right]_{N \times N} = \left[-u_{ji}\right]_{N \times N} = -\mathbf{U}_N^{\mathrm{T}}, \tag{4}$$

the duelist is allowed to legitimately shoot only once at one of the time moments in set T_N. Shooting the bullet is a metaphor for making a single decision, where the duelist benefits from shooting as late as possible but only by shooting first [Alpern & Howard, 2019; Liu et al., 2022; Romanuke, 2024b].

Uniform 1-bullet silent duel (1) by (2)–(4) is a timing game [Barron, 2013; Laraki et al., 2005; Radzik, 1996], in which it is important to find all optimal pure strategies for the duelist to shoot at the best time moment. If there are no optimal pure strategies, where the duel is solved in mixed strategies owing to (1) being a matrix game [Epstein, 2013; Osborne, 2003; Romanuke, 2025a], but an optimal mixed strategy is a practically poor solution because the optimal game value cannot be statistically reached in a single duel round or several rounds [Reinganum, 1989; Romanuke, 2019; Viscolani, 2012].

2. OPTIMAL TIME MOMENT EXISTENCE

As payoff matrix $\mathbf{U}_N$ is skew-symmetric, then duel (1) by (2)–(4) is symmetric [Barron, 2013; Romanuke, 2024a; Romanuke, 2024b]. This means that its optimal game value is 0, and each of the duelists has the same set of optimal strategies, which can be both pure and mixed [Osborne, 2003]. However, due to finite 1-bullet silent duels, like duel (1) by (2)–(4), which commonly intended to model non-repeatable environments, the main goal is to determine all optimal time moments (optimal pure strategies) for the duelist to shoot [Romanuke, 2025a]. If duel (1) is not solved in pure strategies, there are no optimal time moments at the duelist. Then the duel configuration is forcedly modified through changing the structure of payoff matrix $\mathbf{U}_N$ in order to devise a pure-strategy optimal behavior for the duelist [Romanuke, 2024b]. In the case when the duelist possesses two or more optimal time moments (examples of such cases can be found in [Romanuke, 2024c; Romanuke, 2025a]), the selection of the time moment to shoot is based on an additional criterion or criteria [Epstein, 2013; Reinganum, 1989; Romanuke, 2019].

The skew-symmetry of the payoff matrix (4) implies that the main diagonal of this matrix is of N zeros. Another implication is that any saddle point of matrix (4) is a zero entry in a non-negative row and a non-positive column [Osborne, 2003; Romanuke, 2024c; Romanuke, 2025b]. Thus, if row i^* of matrix (4) by $i^* \in \left\{\overline{1, N}\right\}$ is nonnegative, then row i^* contains a saddle point on the main diagonal [Romanuke, 2024a], and time moment t_{i^*} is optimal. A symmetric reasoning is true for columns: if column i^* of matrix (4) by $i^* \in \left\{\overline{1, N}\right\}$ is nonpositive, then column i^* contains a saddle point on the main diagonal [Romanuke, 2024b], and time moment t_{i^*} is optimal. So, optimal time moments can be found by studying only either non-negative rows or non-positive columns of matrix (4). If a row i^* contains only positive entries, except for the main diagonal entry $u_{i^*i^*} = 0$, then there is the single optimal time moment t_{i^*} in duel (1) by (2)–(4) [Romanuke, 2024c; Romanuke, 2025a; Romanuke, 2025b]. If row i^* contains a negative entry (and column i^* contains the positive entry), time moment t_{i^*} is not optimal.

3. SCALED PAYOFF EXPONENTIAL RATE

Obviously, the duelist's optimal behavior in duel (1) by (2)–(4) depends on the structure of the payoff matrix (4), which is determined by how its entries are calculated or determined beforehand [Epstein, 2013; Reinganum, 1989; Romanuke, 2024b]. In general,

$$u_{ij} = ag(x_i) - ag(y_j) + a^2 g(x_i) g(y_j) \operatorname{sign}(y_j - x_i)$$
$$\text{for } i = \overline{1, N} \text{ and } j = \overline{1, N}$$

by some discrete accuracy nondecreasing functions $g(x_i)$ and $g(y_j)$ of the first and second duelists, respectively, scaled with an accuracy factor $a > 0$, where

$$g(x_1) = g(y_1) = g(0) = 0 \text{ and } g(x_N) = g(y_N) = g(1) = 1. \tag{5}$$

Requirements (5) ensure that the accuracy function of the duelist is standardized to a probabilistic interval $[0;1]$ and thus this function may be referred to as a probability of a successful shot. With exponentially increasing accuracy functions, entry u_{ij} of payoff matrix (4) is calculated as

$$u_{ij} = ag\left(e^{x_i}\right) - ag\left(e^{y_j}\right) + a^2 g\left(e^{x_i}\right) g\left(e^{y_j}\right) \operatorname{sign}\left(y_j - x_i\right)$$
$$\text{for } i = \overline{1, N} \text{ and } j = \overline{1, N} \tag{6}$$

by still obeying requirements similar to standardization requirements (5):

$$g\left(e^{x_1}\right) = g\left(e^{y_1}\right) = g\left(e^0\right) = g(1) = 0$$
$$\text{and } g\left(e^{x_N}\right) = g\left(e^{y_N}\right) = g\left(e^1\right) = g(e) = 1, \tag{7}$$

where

$$g\left(e^z\right) = \alpha e^z + \beta \text{ by } \alpha \in \mathbb{R} \setminus \{0\},\ \beta \in \mathbb{R}. \tag{8}$$

As function (8) of variable z must obey requirements (7), then

$$g\left(e^0\right) = g(1) = \alpha + \beta = 0,\ g\left(e^1\right) = g(e) = \alpha e + \beta = 1,$$

whence

$$\beta = -\alpha = 1 - \alpha e,\ \alpha(e-1) = 1,$$

and

$$\alpha = \frac{1}{e-1},\ \beta = \frac{1}{1-e}. \tag{9}$$

Upon plugging (9) into (8), function $g\left(e^z\right)$ becomes an exponentially-convex accuracy function:

$$g\left(e^z\right) = \frac{e^z}{e-1} - \frac{1}{e-1} = \frac{e^z - 1}{e-1}. \tag{10}$$

Then, upon plugging (10) into (6), entry u_{ij} of payoff matrix (4) is calculated as

$$u_{ij} = a \cdot \frac{e^{x_i} - 1}{e-1} - a \cdot \frac{e^{y_j} - 1}{e-1} + a^2 \cdot \frac{e^{x_i} - 1}{e-1} \cdot \frac{e^{y_j} - 1}{e-1} \cdot \operatorname{sign}\left(y_j - x_i\right) =$$
$$= a \cdot \frac{e^{x_i} - e^{y_j}}{e-1} + a^2 \cdot \frac{\left(e^{x_i} - 1\right)\left(e^{y_j} - 1\right)}{(e-1)^2} \cdot \operatorname{sign}\left(y_j - x_i\right)$$
$$\text{for } i = \overline{1, N} \text{ and } j = \overline{1, N}. \tag{11}$$

Discrete surface (11) plots are mosaicked in Figure 1 for a subset of values of the accuracy factor not exceeding 1. As the accuracy factor is increased, a ravine is observable along line $x_i = y_i$ for $i = \overline{1, N}$. So, payoffs for the duelists become more distant, and the duel psychological pressure builds up as the accuracy factor grows. Figure 2 shows a mosaic of discrete surface (11) plots for a subset of values of the accuracy factor above 1, where it is seen that the ravine continues growing deeper and steeper as the accuracy factor is increased further above 1.

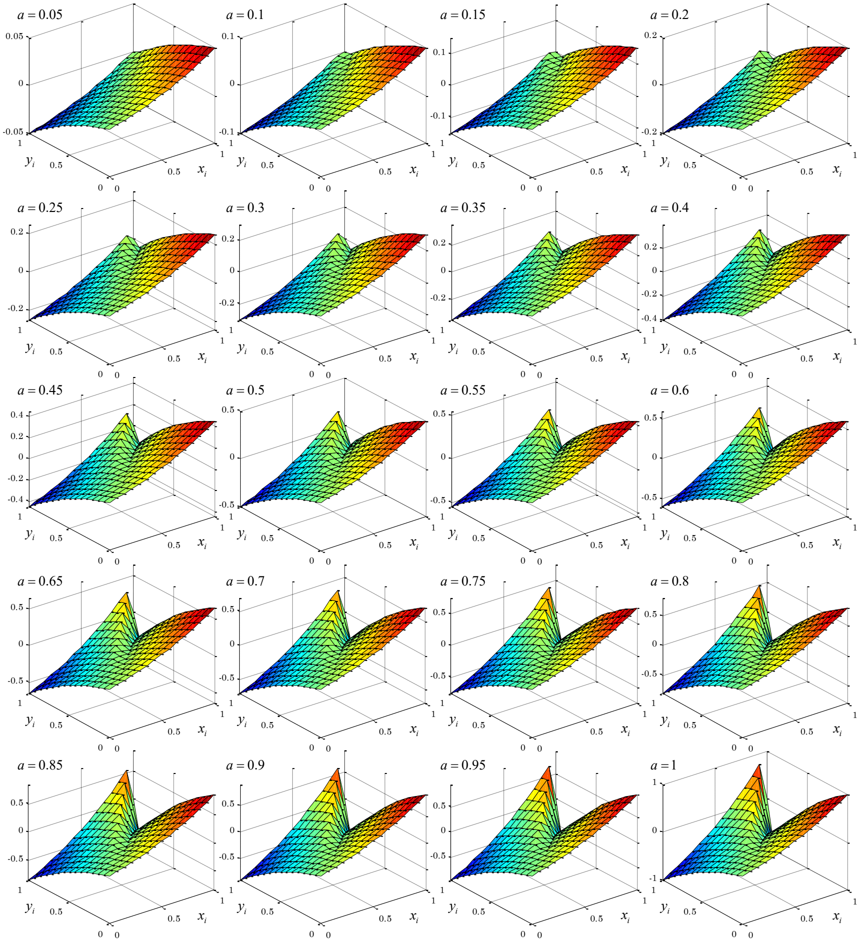

Figure 1. Surface (11) for a subset of values of the accuracy factor not exceeding 1 by $N = 15$

Source: own study.

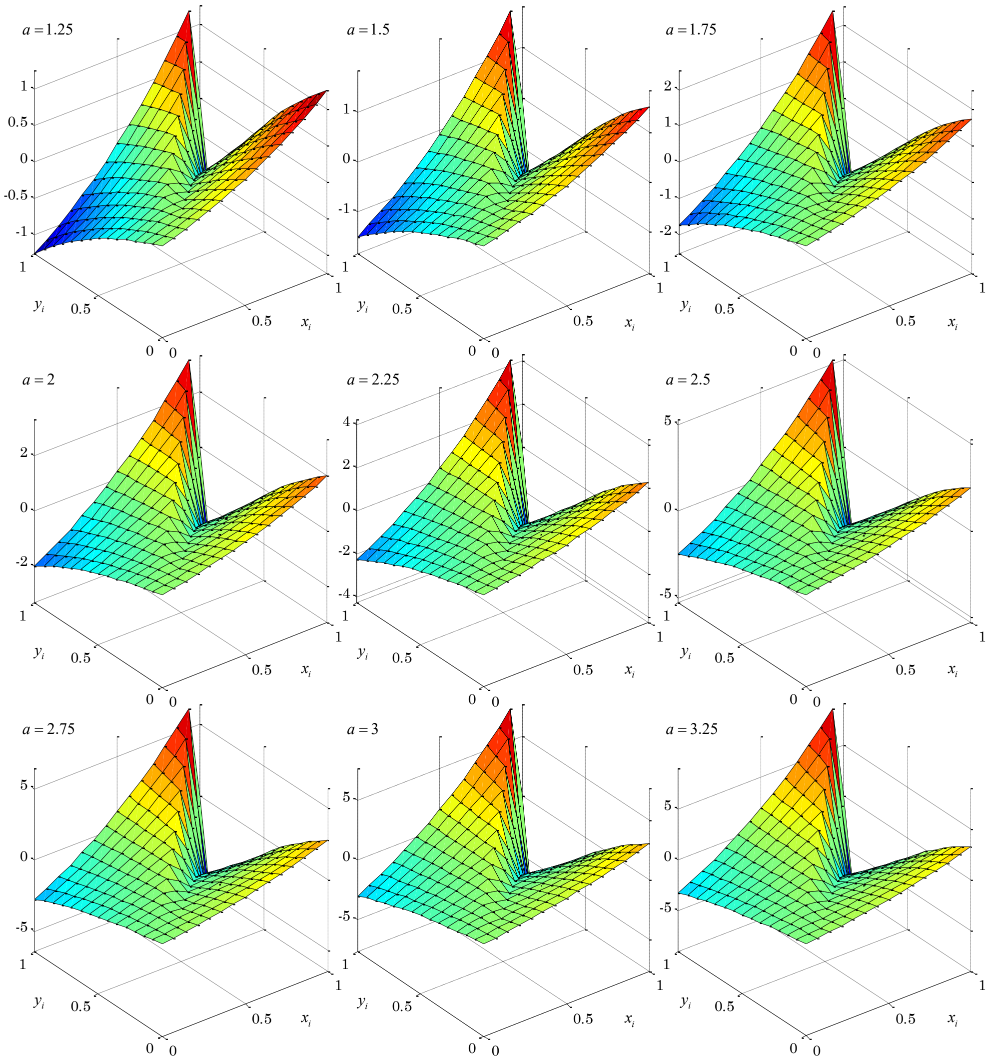

Figure 2. Surface (11) for a subset of values of the accuracy factor above 1 by *N* = 15

Source: own study.

Hence, the goal is to determine optimal time moments for the duelist in uniform 1-bullet silent duel (1) by (2)–(4) and exponentially-convex accuracy payoffs (11). A particular practical interest lies in determining the conditions of optimality of time moments from subset $\left\{t_q\right\}_{q=2}^{N-1}$, i.e., when the very beginning and end of the duel are not accepted as practicable moments (still being the legitimate time moments to shoot, though). Such an interest is typical for economical, jurisprudential, political, and other environments with time constraints, where being too early or too late may violate some ethical standards [Osborne, 2003].

4. OPTIMALITY OF TIME MOMENTS

Theorem 1. The very beginning is never optimal in the uniform 1-bullet silent duel (1) by (2)–(4) and exponentially-convex accuracy payoffs (11).

Proof. Entry

$$u_{1j} = a \cdot \frac{1-e^{y_j}}{e-1} + a^2 \cdot \frac{(1-1)\cdot\left(e^{y_j}-1\right)}{(e-1)^2} = a \cdot \frac{1-e^{y_j}}{e-1} < 0$$
$$\forall\, j = \overline{2, N} \text{ by } N \in \mathbb{N} \setminus \{1, 2\} \tag{12}$$

due to $e^{y_j} > 1 \;\; \forall\, j = \overline{2, N}$. Inequality (12) implies that time moment $t_1 = 0$ is not optimal in the duel.

Lemma 1. Entry u_{nj} by (11), considered as a discrete function of index $j = \overline{1, n-1}$ by $n \in \left\{\overline{2, N}\right\}$, strictly decreases as index j is increased.

Proof. Plugging $i = n$ into (11) for $n \in \left\{\overline{2, N}\right\}$, entry

$$u_{nj} = a \cdot \frac{e^{x_n}-e^{y_j}}{e-1} - a^2 \cdot \frac{\left(e^{x_n}-1\right)\left(e^{y_j}-1\right)}{(e-1)^2} =$$
$$= a \cdot \frac{e^{x_n}-1}{e-1} - a^2 \cdot \left(\frac{1}{a} + \frac{e^{x_n}-1}{e-1}\right) \cdot \frac{e^{y_j}-1}{e-1} \text{ for } j = \overline{1, n-1} \text{ at } n \in \left\{\overline{2, N}\right\}. \tag{13}$$

Due to $e^{x_n} > 1$ and $e^{y_j} \geqslant 1$ by $x_n > 0$ and $y_j \geqslant 0$, respectively, entry (13) is a negatively sloped line with respect to exponent e^{y_j}. Therefore, entry (13) strictly decreases as index j is increased from 1 up to $n-1$.

Lemma 2. Entry u_{nj} by (11), considered as a discrete function of index $j = \overline{n+1, N}$ by $n \in \left\{\overline{2, N-1}\right\}$, strictly decreases as the index j is increased by

$$a \in \left(0; \frac{e-1}{e^{x_n}-1}\right). \tag{14}$$

At

$$a = \frac{e-1}{e^{x_n}-1} \tag{15}$$

entry

$$u_{nj} = 1 \quad \forall\, j = \overline{n+1, N}\,. \tag{16}$$

Entry u_{nj} strictly increases as index j is increased by

$$a > \frac{e-1}{e^{x_n}-1}\,. \tag{17}$$

Proof. Plugging $i = n$ into (11) for $n \in \left\{\overline{2, N-1}\right\}$, entry

$$u_{nj} = a \cdot \frac{e^{x_n} - e^{y_j}}{e-1} + a^2 \cdot \frac{\left(e^{x_n}-1\right)\left(e^{y_j}-1\right)}{(e-1)^2} =$$

$$= a \cdot \frac{e^{x_n}-1}{e-1} - a^2 \cdot \left(\frac{1}{a} - \frac{e^{x_n}-1}{e-1}\right) \cdot \frac{e^{y_j}-1}{e-1} \text{ for } j = \overline{n+1, N} \text{ at } n \in \left\{\overline{2, N-1}\right\}. \tag{18}$$

Due to $e^{x_n} > 1$ and $e^{y_j} > 1$ by $x_n > 0$ and $y_j > 0$, respectively, entry (18) is a negatively sloped line with respect to exponent e^{y_j} if

$$\frac{1}{a} - \frac{e^{x_n}-1}{e-1} > 0\,. \tag{19}$$

Inequality (19) is equivalent to condition (14), by which entry (18) strictly decreases as index j is increased off $n+1$ up to N. If (15) is true, entry (18) becomes:

$$u_{nj} = a \cdot \frac{e^{x_n}-1}{e-1} - a^2 \cdot \left(\frac{e^{x_n}-1}{e-1} - \frac{e^{x_n}-1}{e-1}\right) \cdot \frac{e^{y_j}-1}{e-1} =$$

$$= a \cdot \frac{e^{x_n}-1}{e-1} = \frac{e-1}{e^{x_n}-1} \cdot \frac{e^{x_n}-1}{e-1} = 1,$$

which confirms (16). If inequality (17) holds, entry (18) becomes a positively sloped line with respect to exponent e^{y_j}, i.e., the entry strictly increases as index j is increased off $n+1$ up to N.

Theorem 2. In the uniform 1-bullet silent duel (1) by (2)–(4) and exponentially--convex accuracy payoffs (11), time moment

$$t_n = \frac{n-1}{N-1} \text{ for some } n \in \left\{\overline{2,\, N-1}\right\} \text{ by } N \in \mathbb{N} \setminus \{1,\, 2\} \tag{20}$$

is optimal if

$$n \in \left[1 + (N-1)\ln\frac{e+a}{1+a};\, \upsilon\right] \bigcap \left\{\overline{2,\, N-1}\right\} \tag{21}$$

by

$$\upsilon = 1 + (N-1)\ln\left(\frac{e^{\frac{1}{N-1}} \cdot (e+a-1) - e + a + 1}{2a} + \right.$$

$$\left. + \frac{1}{2a}\sqrt{\left((e-a-1) - e^{\frac{1}{N-1}} \cdot (e+a-1)\right)^2 - 4a^2 e^{\frac{1}{N-1}}}\right) \tag{22}$$

and condition (14). Besides, time moment (20) is optimal if

$$n \in [2;\, \upsilon] \bigcap \left\{\overline{2,\, N-1}\right\} \tag{23}$$

by (22) and condition (15). Time moment (20) is also optimal if

$$n \in [\omega;\, \upsilon] \bigcap \left\{\overline{2,\, N-1}\right\} \tag{24}$$

by

$$\omega = 1 + (N-1)\ln\left(\frac{a + e - 1 - e^{-\frac{1}{N-1}} \cdot (e-1-a)}{2a} + \right.$$

$$\left. + \frac{1}{2a}\sqrt{\left((1-a-e) + e^{-\frac{1}{N-1}} \cdot (e-1-a)\right)^2 - 4a^2 e^{-\frac{1}{N-1}}}\right) \tag{25}$$

and (22) and inequality (17).

Proof. Time moment (20) is optimal in an $N \times N$ duel if inequalities

$$u_{nj} = a \cdot \frac{e^{x_n} - e^{y_j}}{e-1} - a^2 \cdot \frac{\left(e^{x_n} - 1\right)\left(e^{y_j} - 1\right)}{(e-1)^2} \geqslant 0 \quad \forall\, y_j < x_n \text{ by } j = \overline{1, n-1} \tag{26}$$

and

$$u_{nj} = a \cdot \frac{e^{x_n} - e^{y_j}}{e-1} + a^2 \cdot \frac{\left(e^{x_n} - 1\right)\left(e^{y_j} - 1\right)}{(e-1)^2} \geqslant 0 \quad \forall\, y_j > x_n \text{ by } j = \overline{n+1, N} \tag{27}$$

hold. The monotonicity conditions from *Lemmas 1* and *2* allow simplifying inequalities (26) and (27) by the removal of index j. Owing to *Lemma 1*, inequality (26) is equivalent to inequality

$$u_{n,n-1} = a \cdot \frac{e^{x_n} - e^{y_{n-1}}}{e-1} - a^2 \cdot \frac{\left(e^{x_n} - 1\right)\left(e^{y_{n-1}} - 1\right)}{(e-1)^2} \geqslant 0\,. \tag{28}$$

In its turn, inequality (28), upon having removed the positive denominator $(e-1)^2$, is equivalent to inequality

$$(e-1)\left(e^{x_n} - e^{y_{n-1}}\right) - a\left(e^{x_n} - 1\right)\left(e^{y_{n-1}} - 1\right) =$$

$$= e \cdot e^{x_n} - e^{x_n} - e \cdot e^{y_{n-1}} + e^{y_{n-1}} - ae^{x_n} \cdot e^{y_{n-1}} + ae^{y_{n-1}} + ae^{x_n} - a =$$

$$= e^{x_n} \cdot (e - 1 + a) + e^{y_{n-1}} \cdot (1 + a - e) - ae^{x_n} \cdot e^{y_{n-1}} - a \geqslant 0\,,$$

whence, upon having plugged in the respective elements of sets (2) and (3), inequality

$$e^{\frac{n-1}{N-1}} \cdot (e - 1 + a) + e^{\frac{n-2}{N-1}} \cdot (1 + a - e) - ae^{\frac{n-1}{N-1}} \cdot e^{\frac{n-2}{N-1}} - a \geqslant 0 \tag{29}$$

emerges. To solve inequality (29), denote

$$b = e^{\frac{n-1}{N-1}}\,. \tag{30}$$

Then

$$e^{\frac{n-2}{N-1}} = b \cdot e^{-\frac{1}{N-1}}$$

and thus inequality (29) is represented as

$$b \cdot (e - 1 + a) + b \cdot e^{-\frac{1}{N-1}} \cdot (1 + a - e) - ab \cdot b \cdot e^{-\frac{1}{N-1}} - a \geqslant 0$$

or

$$e^{-\frac{1}{N-1}} \cdot ab^2 + b \cdot \left(e^{-\frac{1}{N-1}} \cdot (e-a-1) - e - a + 1 \right) + a \leqslant 0 . \qquad (31)$$

Inequality (31) is quadratic with respect to variable (30), where the accuracy factor a and duel size N can be considered as parameters of quadratic inequality (31). The discriminant of the respective quadratic equation

$$e^{-\frac{1}{N-1}} \cdot ab^2 + b \cdot \left(e^{-\frac{1}{N-1}} \cdot (e-a-1) - e - a + 1 \right) + a = 0$$

is

$$D = \left(e^{-\frac{1}{N-1}} \cdot (e-a-1) - e - a + 1 \right)^2 - 4a^2 e^{-\frac{1}{N-1}} ,$$

and so inequality (31) holds by $b \in [b_1; b_2]$, where

$$b_1 = \frac{ -\left(e^{-\frac{1}{N-1}} \cdot (e-a-1) - e - a + 1 \right) - \sqrt{ \left(e^{-\frac{1}{N-1}} \cdot (e-a-1) - e - a + 1 \right)^2 - 4a^2 e^{-\frac{1}{N-1}} } }{ 2ae^{-\frac{1}{N-1}} } =$$

$$= \frac{ e^{\frac{1}{N-1}} \cdot (e+a-1) - e + a + 1 }{2a} - \frac{1}{2a} \sqrt{ \left((e-a-1) - e^{\frac{1}{N-1}} \cdot (e+a-1) \right)^2 - 4a^2 e^{\frac{1}{N-1}} } ,$$

where for intermediate simplification it is used that

$$\left(e^{-\frac{1}{N-1}} \cdot (e-a-1) - e - a + 1 \right)^2 \cdot e^{\frac{1}{N-1}} \cdot e^{\frac{1}{N-1}} =$$

$$= \left[\left(e^{-\frac{1}{N-1}} \cdot (e-a-1) - e - a + 1 \right) \cdot e^{\frac{1}{N-1}} \right]^2 =$$

$$= \left((e-a-1) - e^{\frac{1}{N-1}} \cdot (e+a-1) \right)^2 ,$$

and

$$b_2 = \frac{e^{\frac{1}{N-1}}\cdot(e+a-1)-e+a+1}{2a} + \\ + \frac{1}{2a}\sqrt{\left((e-a-1)-e^{\frac{1}{N-1}}\cdot(e+a-1)\right)^2-4a^2e^{\frac{1}{N-1}}}\,. \tag{32}$$

Using denotation (30), root (32) is

$$b_2 = e^{\frac{\upsilon-1}{N-1}} = \frac{e^{\frac{1}{N-1}}\cdot(e+a-1)-e+a+1}{2a} + \\ + \frac{1}{2a}\sqrt{\left((e-a-1)-e^{\frac{1}{N-1}}\cdot(e+a-1)\right)^2-4a^2e^{\frac{1}{N-1}}} \text{ for some } \upsilon \in \left\{\overline{2,\, N-1}\right\},$$

whence

$$\frac{\upsilon-1}{N-1} = \ln\left(\frac{e^{\frac{1}{N-1}}\cdot(e+a-1)-e+a+1}{2a} + \right. \\ \left. + \frac{1}{2a}\sqrt{\left((e-a-1)-e^{\frac{1}{N-1}}\cdot(e+a-1)\right)^2-4a^2e^{\frac{1}{N-1}}}\right)$$

and thus expression (22) for the right endpoint in closed interval (21) emerges. Here, however,

$$b_1 b_2 = \frac{a}{e^{-\frac{1}{N-1}}\cdot a} = e^{\frac{1}{N-1}},$$

whence

$$b_1 = \frac{e^{\frac{1}{N-1}}}{b_2} = e^{\frac{1}{N-1}}\cdot e^{\frac{1-\upsilon}{N-1}} = e^{\frac{2-\upsilon}{N-1}} \leqslant 1 \text{ for some } \upsilon \in \left\{\overline{2,\, N-1}\right\},$$

which means that inequality (31) holds just by $b \leqslant b_2$ owing to $b > 1$ by (30), where root b_1 can be ignored. So, inequality (26) holds just if membership (23) for (22) is true.

Owing to *Lemma 2*, inequality (27) by condition (14) is equivalent to inequality

$$u_{nN} = a \cdot \frac{e^{x_n} - e^{y_N}}{e-1} + a^2 \cdot \frac{\left(e^{x_n} - 1\right)\left(e^{y_N} - 1\right)}{(e-1)^2} \geqslant 0 . \tag{33}$$

Inequality (33) is simplified to inequality

$$\frac{e^{x_n} - e}{e-1} + a \cdot \frac{\left(e^{x_n} - 1\right)(e-1)}{(e-1)^2} =$$

$$= \frac{e^{x_n} - e}{e-1} + \frac{a \cdot \left(e^{x_n} - 1\right)}{e-1} = \frac{e^{x_n} \cdot (1+a) - e - a}{e-1} \geqslant 0 ,$$

whence, upon having removed the positive denominator $e-1$, the sequence of simplified inequalities emerges:

$$e^{\frac{n-1}{N-1}} \cdot (1+a) - e - a \geqslant 0 ,$$

$$e^{\frac{n-1}{N-1}} \geqslant \frac{e+a}{1+a} ,$$

$$\frac{n-1}{N-1} \geqslant \ln \frac{e+a}{1+a} ,$$

$$n \geqslant 1 + (N-1) \ln \frac{e+a}{1+a} . \tag{34}$$

So, time moment (20) is optimal if inequality $n \leqslant \upsilon$ by (22) and $n \in \left\{\overline{2, N-1}\right\}$ holds along with inequality (34) by (14), i.e., membership (21) is true by (14) and (22). If (15) is true, then (16) holds owing to *Lemma 2*, whence inequality (27) holds outright. Therefore, time moment (20) is optimal by (15) if just membership (23) for (22) is true.

If inequality (17) is true, then, owing to *Lemma 2*, inequality (27) is equivalent to inequality

$$u_{n,n+1} = a \cdot \frac{e^{x_n} - e^{y_{n+1}}}{e-1} + a^2 \cdot \frac{\left(e^{x_n} - 1\right)\left(e^{y_{n+1}} - 1\right)}{(e-1)^2} \geqslant 0 . \tag{35}$$

Inequality (35), upon having removed the positive denominator $(e-1)^2$, is simplified to inequality

$$(e-1)\left(e^{x_n}-e^{y_{n+1}}\right)+a\left(e^{x_n}-1\right)\left(e^{y_{n+1}}-1\right)=$$
$$=e\cdot e^{x_n}-e^{x_n}-e\cdot e^{y_{n+1}}+e^{y_{n+1}}+ae^{x_n}\cdot e^{y_{n+1}}-ae^{y_{n+1}}-ae^{x_n}+a=$$
$$=e^{x_n}\cdot(e-1-a)+e^{y_{n+1}}\cdot(1-a-e)+ae^{x_n}\cdot e^{y_{n+1}}+a\geqslant 0,$$

whence

$$e^{\frac{n-1}{N-1}}\cdot(e-1-a)+e^{\frac{n}{N-1}}\cdot(1-a-e)+ae^{\frac{n-1}{N-1}}\cdot e^{\frac{n}{N-1}}+a\geqslant 0\,. \tag{36}$$

Using denotation (30),

$$e^{\frac{n}{N-1}}=b\cdot e^{\frac{1}{N-1}}$$

and thus inequality (36) is represented as

$$b\cdot(e-1-a)+b\cdot e^{\frac{1}{N-1}}\cdot(1-a-e)+ab\cdot b\cdot e^{\frac{1}{N-1}}+a\geqslant 0\,. \tag{37}$$

Inequality (37) is quadratic with respect to variable (30) by accuracy factor a and duel size N as parameters. The discriminant of the respective quadratic equation

$$e^{\frac{1}{N-1}}\cdot ab^2+b\cdot\left(e^{\frac{1}{N-1}}\cdot(1-a-e)+e-1-a\right)+a=0$$

is

$$D=\left(e^{\frac{1}{N-1}}\cdot(1-a-e)+e-1-a\right)^2-4a^2e^{\frac{1}{N-1}},$$

and so inequality (37) holds by $b\in(0;b_3]\bigcup[b_4;e)$, where

$$b_3=\frac{-\left(e^{\frac{1}{N-1}}\cdot(1-a-e)+e-1-a\right)-\sqrt{\left(e^{\frac{1}{N-1}}\cdot(1-a-e)+e-1-a\right)^2-4a^2e^{\frac{1}{N-1}}}}{2ae^{\frac{1}{N-1}}}=$$
$$=\frac{-\left(1-a-e+e^{-\frac{1}{N-1}}\cdot(e-1-a)\right)}{2a}-$$
$$-\frac{1}{2a}\sqrt{\left((1-a-e)+e^{-\frac{1}{N-1}}\cdot(e-1-a)\right)^2-4a^2e^{-\frac{1}{N-1}}}\,, \tag{38}$$

where for intermediate simplification it is used that

$$\left(e^{\frac{1}{N-1}}\cdot(1-a-e)+e-1-a\right)^2\cdot e^{-\frac{1}{N-1}}\cdot e^{-\frac{1}{N-1}}=$$

$$=\left[\left(e^{\frac{1}{N-1}}\cdot(1-a-e)+e-1-a\right)\cdot e^{-\frac{1}{N-1}}\right]^2=\left((1-a-e)+e^{-\frac{1}{N-1}}\cdot(e-1-a)\right)^2,$$

and

$$b_4=e^{\frac{\omega-1}{N-1}}=\frac{-\left(1-a-e+e^{-\frac{1}{N-1}}\cdot(e-1-a)\right)}{2a}+$$

$$+\frac{1}{2a}\sqrt{\left((1-a-e)+e^{-\frac{1}{N-1}}\cdot(e-1-a)\right)^2-4a^2e^{-\frac{1}{N-1}}}\ \text{for some}\ \omega\in\left\{\overline{2,\,N-1}\right\}.\quad(39)$$

From root (39), it follows that

$$\frac{\omega-1}{N-1}=\ln\left(\frac{a+e-1-e^{-\frac{1}{N-1}}\cdot(e-1-a)}{2a}+\right.$$

$$\left.+\frac{1}{2a}\sqrt{\left((1-a-e)+e^{-\frac{1}{N-1}}\cdot(e-1-a)\right)^2-4a^2e^{-\frac{1}{N-1}}}\right)$$

and thus expression (25) emerges. Here, however,

$$b_3b_4=\frac{a}{e^{\frac{1}{N-1}}\cdot a}=e^{-\frac{1}{N-1}},$$

whence

$$b_3=\frac{e^{-\frac{1}{N-1}}}{b_4}=e^{-\frac{1}{N-1}}\cdot e^{\frac{1-\omega}{N-1}}=e^{-\frac{\omega}{N-1}}<1\ \text{for some}\ \omega\in\left\{\overline{2,\,N-1}\right\},$$

which means that inequality (37) holds just by $b\geqslant b_4$ owing to $b>1$ by (30), where the root b_3 can be ignored. So, inequality (27) holds just if membership

$$n \in [\omega; N-1] \cap \left\{\overline{2, N-1}\right\} \tag{40}$$

for (25) is true. Hence, time moment (20) by (17) is optimal if inequality $n \leqslant \upsilon$ by (22) and $n \in \left\{\overline{2, N-1}\right\}$ holds along with membership (40), i.e., membership (24) is true by (17) and (25), (22).

Consider a numerical example of applying *Theorem 2*. If, for instance, $N = 5$ and $a = 1$, then condition (14) holds and the right endpoint in closed interval (21) is

$$\upsilon = 1 + 4 \cdot \ln\left(\frac{e^{\frac{5}{4}} - e + 2}{2} + \frac{1}{2}\sqrt{\left((e-2) - e^{\frac{5}{4}}\right)^2 - 4e^{\frac{1}{4}}} \right) \approx 4.125 \text{ (a } 5 \cdot 10^{-4} \text{ roundoff),}$$

whereas the left endpoint in the closed interval (21) is

$$1 + 4 \cdot \ln\frac{e+1}{2} \approx 3.4805 \text{ (a } 5 \cdot 10^{-4} \text{ roundoff).}$$

Hence, time moment (20) is optimal if (21) holds, i.e., an integer n belongs to the interval between 3.4805 and 4.125, which is only possible for $n = 4$. So, time moment $t_4 = \frac{3}{4}$ is optimal in the 5×5 duel by $a = 1$. The existence or non-existence of time moments is verified by *Theorem 2* for any other parameters N and a in similar manner.

5. CONCLUSIONS

In the uniform 1-bullet silent duel (1) by (2)–(4) and exponentially-convex accuracy payoffs (11), an optimal time moment for the duelist exists under the conditions stated in *Theorem 2*. Numerical experiments confirm that the optimality can be manipulated by changing the accuracy factor a, although its generalized impact has yet to be determined. The obtained results are applicable in systems under limited or censored communication with uncertainty, latency, and lucrative delayed actions [Ewerhart, 2020; Reinganum, 1989; Romanuke, 2024c; Romanuke, 2025b]. The examples of such set-ups are, for example, time-sensitive information release (privacy and censorship), queueing and load balancing (information science and telecommunication systems), and block proposal timing for decentralized consensus protocols (in Proof-of-Work and Proof-of-Stake) [Gans, 2023; Wang & Wu, 2025].

The influence of the accuracy factor on the optimal moment position is nonlinear. However, in specific possible applications of the duel model, the developer may change parameters N and a so that the optimal moment would exist after 0 and before 1. Thus, in blockchain consensus systems with time-dependent rewards, block proposal constitutes a single irreversible action within a discretized time window. Rewards typically increase over time due to transaction accumulation or improved network state, while the risk of rejection rises near the deadline due to latency and contention. When proposals are not observable by competitors until the round ends, validators face a silent competitive timing problem. Progressive reward growth then must induce an equilibrium proposal time located late in the window but strictly before its end, balancing reward amplification against preemption risk [Schwarz-Schilling et al., 2023].

A similar timing structure appears in commit-reveal schemes, decentralized oracles, and sealed-bid mechanisms. Participants may reveal information only once, the value of which increases with delay but collapses if a competitor reveals it first. Since actions remain unobserved until the reveal phase concludes, strategic interaction occurs under informational silence. With progressively increasing payoffs, optimal behavior must concentrate at a late but interior reveal moment that ensures informational maturity without excessive exposure to being overtaken [Moallemi et al., 2025].

More broadly, blockchain protocols with enforced silence or delayed validation phases transform timing into a strategic choice under uncertainty. When payoff functions grow progressively over time while penalties escalate near the end of the horizon, the resulting equilibrium timing must become uniquely determined [Robin, 2025]. Without explicit coordination, agents by the means of an appropriate couple $\{N, a\}$ by *Theorem 2* are implicitly guided (or, speaking openly, latently forced) toward a stable interior action moment, demonstrating how uniform reward scheduling can serve as a mechanism-design tool for controlling strategic timing in decentralized systems.

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