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A NOTE ON TREATMENT OF SIGNALS IN DIGITAL SIGNAL PROCESSING AND IN NETWORK CALCULUS

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Abstract: This short paper presents from the perspective of the operator theory some basic operations performed on signals in the digital signal processing as well as in the network calculus. These are the following operations: signal sampling, amplitude quantization, and signal recovery from its samples – in the digital signal processing. And regarding the network calculus, building up an auxiliary (continuous) traffic flow and recovery of a real traffic that possesses a non-continuous structure (with some granularity) after manipulations that were carried out with the use of a flow model are discussed in this paper. Some interesting results achieved and interpretations regarding the aforementioned stuff are presented.

Keywords: basic operations on signals in digital signal processing and in network calculus, application of operator theory, inverse operators.

1. INTRODUCTION

In this paper, some interesting results regarding basic operations performed on signals in the digital signal processing [Tse and Viswanath 2005; Nurfaizey et al. 2012] as well as in the network calculus a[Landman et al. 2008; Szabatin 2016] are presented. They have been derived with the use of the operator theory and deal with the following operations: signal sampling, amplitude quantization, and signal recovery from its samples in case of the digital signal processing. Furthermore, in the second case, they regard the building up an auxiliary (continuous) traffic flow and recovery of a real traffic that possesses a non-continuous structure (with some granularity) from that traffic, which was obtained in calculations based on the use of a flow model. A role played by the corresponding inverse operations and operators in treatment of the above stuff was highlighted.

2. SIGNALS IN SIGNAL PROCESSING AND IN NETWORK CALCULUS

Signals are defined differently in signal processing and in network calculus. In both cases, however, their variants defined for a continuous time as well as those being functions of a discrete time variable are used.

Let us start with the signals which are used in the signal processing. To this end, let $x(t)$ be a signal defined for a continuous time variable *t*. Further, when this signal has to be processed digitally, it is sampled in time and its instantaneous amplitude is quantized. As a result we obtain then

$$\begin{aligned} x_d[n] = x_d[nT] = \boldsymbol{H}_Q\left(\boldsymbol{H}_S\left(x(t)\right)\right) = \\ = \left(\boldsymbol{H}_Q \boldsymbol{H}_S\right)(x)(t) = \left(\boldsymbol{H}_D\right)(x)(t) \end{aligned} \tag{1}$$

where $x_d[n]$ or $x_d[nT]$ means the discrete version of $x(t)$ (that is a one which is sampled in time and quantized in amplitude), *n* denotes an index of a sampling moment, while *T* a sampling period. Furthermore, $\boldsymbol{H}_S$ and $\boldsymbol{H}_Q$ mean an operator performing the operation of sampling in time and an operator doing a signal amplitude quantization, respectively. The definition of a signal discretization operator $\boldsymbol{H}_D$ (that performs both kinds of discretization: in time and in amplitude) is given in the second row of (1). This is a composite operator $\boldsymbol{H}_D = \boldsymbol{H}_Q \boldsymbol{H}_S$.

In this paper, the result of applying the operator $\boldsymbol{H}_S$ to the signal $x(t)$ is understood as a function $x_s[nT] = x_s[n]$ of a discrete time variable *nT* (expressed in short by an index *n* of this time moment) as illustrated in Fig. 1. Moreover, the following: $x_s[nT] = x(t = nT)$ holds.

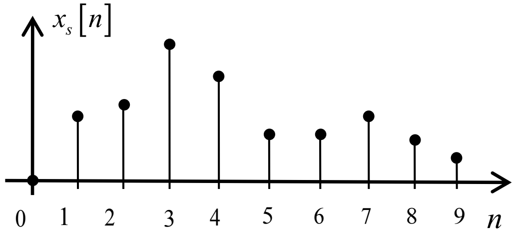

Fig. 1. Illustration to the operation performed by the operator $\boldsymbol{H}_S$

Note that in the area of digital signal processing an operation called a "sample and hold" (that samples a signal in time, and holds its corresponding sampled value during the whole sampling period T) is also used. This operation can be modeled as a concatenation of two operations: a one which is a sampling in time and the second being a holding a sampled amplitude value. That is by the previously introduced operator $\boldsymbol{H}_S$ together with a "holding" operator $\boldsymbol{H}_H$, which we are introducing now in a descriptive way in Fig. 2.

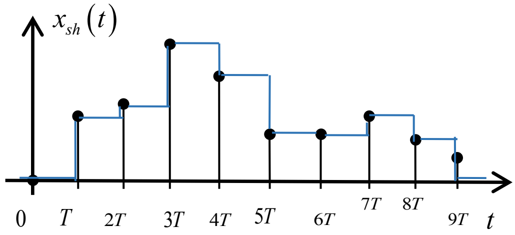

Fig. 2. Illustration to the operation performed by the concatenation of operators $\boldsymbol{H}_S$ and $\boldsymbol{H}_H$

We see from Fig. 2 that the "sample and hold" signal $x_{sh}(t)$, defined by

$$x_{sh}(t) = \boldsymbol{H}_H\left(\boldsymbol{H}_S\left(x(t)\right)\right) = \left(\boldsymbol{H}_H\boldsymbol{H}_S\right)(x)(t) = \\ = \boldsymbol{H}_H\left(x_s[nT]\right) \tag{2}$$

is a signal of a continuous time variable t – in contrast to $x_s[nT] = x_s[n]$.

It is visualized in Fig. 2 by a blue line.

Always, a crucial point in the digital signal processing is recovering the original signal $x(t)$ from its discrete counterpart $x_d[n]$. That is a reconstruction of $x(t)$ provided $x_d[n]$ is given. With the use of the operator terminology introduced in this section, we can view the reconstruction process as searching for the inverses of the operators defined above. This is expressed in what follows.

Note that from (1) we get

$$x(t) = \boldsymbol{H}_D^{-1}\left(x_d[n]\right) = \boldsymbol{H}_S^{-1}\left(\boldsymbol{H}_Q^{-1}\left(x_d[n]\right)\right) = \\ = \left(\boldsymbol{H}_S^{-1}\boldsymbol{H}_Q^{-1}\right)\left(x_d[n]\right) = \left(\boldsymbol{H}_S^{-1}\boldsymbol{H}_Q^{-1}\right)\left(x_d\right)[n] \tag{3}$$

where $\boldsymbol{H}_D^{-1}$, $\boldsymbol{H}_S^{-1}$ and $\boldsymbol{H}_Q^{-1}$ are the corresponding inverse operators of $\boldsymbol{H}_D$, $\boldsymbol{H}_S$, $\boldsymbol{H}_Q$, respectively.

It follows also from (3) and the definition of the operator $\boldsymbol{H}_S$ that reversing the operation performed by $\boldsymbol{H}_Q$ results in

$$x_s\left[nT\right]=\boldsymbol{H}_Q^{-1}\left(x_d\left[nT\right]\right). \tag{4}$$

Further, introduce now an operator $\boldsymbol{H}_F$ that carries out a low-pass filtering of the signals $x_{sh}\left(t\right)$ such that

$$x\left(t\right)=\boldsymbol{H}_F\left(x_{sh}\left(t\right)\right) \tag{5}$$

holds. And in the next step, note that by applying (2) and (4) in (5), we obtain

$$\begin{aligned} x\left(t\right)&=\boldsymbol{H}_F\left(\boldsymbol{H}_H\left(\left(\boldsymbol{H}_Q^{-1}\left(x_d\left[nT\right]\right)\right)\right)\right)= \\ &=\left(\boldsymbol{H}_F\boldsymbol{H}_H\boldsymbol{H}_Q^{-1}\right)\left(x_d\right)\left[nT\right]\ . \end{aligned} \tag{6}$$

Comparison of (6) with (3) shows that

$$\boldsymbol{H}_F\boldsymbol{H}_H=\boldsymbol{H}_S^{-1}\ \Rightarrow\ \boldsymbol{H}_S^{-1}=\boldsymbol{H}_F\boldsymbol{H}_H\ . \tag{7}$$

This means that the inverse operation $\boldsymbol{H}_S^{-1}$ recovering the signal $x\left(t\right)$ from its samples $x_s\left[nT\right]=x_s\left[n\right]$ can be realized as a composite operation comprising of "holding a sampled signal amplitude" and low pass filtering operations. As we know, in such a way, it is realized in practice.

Theory and practice show that there are cases when the aforementioned inverse operators $\boldsymbol{H}_S^{-1}$ and $\boldsymbol{H}_Q^{-1}$ as well as the composite one $\boldsymbol{H}_D^{-1}$ are ill-defined or do not exist. Then, where appropriate, they can be replaced by some kind of pseudo-inverses associated with them. However, this is a topic, in fact, for another paper and because of this not discussed here.

Let us consider the above problem in more detail and start our discussion with the reverse quantization operator $\boldsymbol{H}_Q^{-1}$. To this end, note that according to the Widrow's linear quantization model we can write

$$x_s\left[n\right]+e\left[n\right]=x_d\left[n\right]\ \Rightarrow\ x_s\left[n\right]=x_d\left[n\right]-e\left[n\right]\ , \tag{8}$$

where $e\left[n\right]$ means an error signal to the sampled signal $x_s\left[n\right]$. Further, it is assumed in the Widrows's model that the error signal represents a stochastic process with a uniform distribution. Furthermore, because of the form of (8) this model is called an additive one.

Now, note that comparison of (8) with (4) allows us to write

$$\boldsymbol{H}_Q^{-1}\left(x_d\left[nT\right]\right)=x_d\left[nT\right]-e\left[nT\right] \tag{9}$$

which at the same time gives us the definition of the reverse operator $\boldsymbol{H}_Q^{-1}$. Obviously, because of the nature of its component $e\left[n\right]$ it is a stochastic operator. And by virtue of this, it is not able to recover the deterministic values $x_s\left[n\right]$ provided the $x_d\left[nT\right]$ ones are given.

So, because of the above reason the problem of the reversibility of the operator $\boldsymbol{H}_Q$ was defined in the literature in another way. First of all, it has been limited to consideration of the stochastic signals $x(t)$ only what means, by virtue of this assumption, that the values of $x_d\left[nT\right]$ in (9) are random in this case. So, in this context, the problem was formulated as searching for conditions under which the probability density function (PDF) of a random variable $x(t)$ with t fixed can be fully recovered from their corresponding random variable x_d. Additionally, it has been also assumed that the PDFs of all the random variables $x(t)$ or $x_s\left[nT\right]$ for different values of t or nT are the same. With this, the problem stated above was reduced to consideration of the recoverability of only one random variable, say X or X_s. And this was not done in the literature [Zoelzer 2008] in a direct way, but by exploiting the so-called characteristic functions of the PDFs of the random variables X or X_s and its discrete X_d counterpart.

Let us denote these characteristic functions, similarly as $P_X\left(u\right)$ and $P_{X_d}\left(u\right)$. Further, we recall also here that they are the Fourier transforms (more precisely, the inverse Fourier transforms) of the PDFs of the corresponding random variables. That is they are given by the following relations

$$P_X\left(u\right)=\int_{-\infty}^{\infty} p_X\left(x\right)\exp\left(j2\pi ux\right)dx \tag{10a}$$

and

$$P_{X_d}\left(u\right)=\int_{-\infty}^{\infty} p_{X_d}\left(x_d\right)\exp\left(j2\pi ux_d\right)dx_d\,, \tag{10b}$$

respectively. In (10a) and (10b), $p_X\left(x\right)$ and $p_{X_d}\left(x_d\right)$ are the PDFs of the random variables X and X_d, respectively. Moreover, $j=\sqrt{-1}$, but the variable u is called a quantization frequency. Note also that from the previous assumptions it follows that $p_X\left(x\right)=p_{X_s}\left(x_s\right)$ and $P_X\left(u\right)=P_{X_s}\left(u\right)$ as well.

It has been shown in [Zoelzer 2008] that the following relation

$$P_{X_d}\left(u\right)=\sum_{k=-\infty}^{\infty}P_X\left(u-ku_0\right)\frac{\sin\left(\pi Q\left(u-ku_0\right)\right)}{\pi Q\left(u-ku_0\right)} \quad , \tag{11}$$

holds. In (11), Q means the quantization step.

From (11), the so-called Widrow's quantization theorem can be deduced. It says that if the value of the quantization frequency $u_0 = 1/Q$ occurring in (11) is greater than $2 \cdot u_{\max}$, where $u_{\max}$ denotes the value of the highest quantization frequency occurring in the characteristic function $P_X\left(u\right)$, then the spectra which recur periodically do not overlap. This means that then the perfect reconstruction of $p_X\left(x\right)$ from its quantized $p_{X_d}\left(x_d\right)$ is possible.

In summary, note that the Widrow's quantization theorem with the relation (11) describe the conditions and form of the inverse operator $\boldsymbol{H}_Q^{-1}$, meant as a stochastic one, in terms of the characteristic functions of the random variables $X$and X_d. Further, observe also that in the case of assuming the signal $x_d\left[nT\right]$ to be a deterministic one, we get from (9) a deterministic approximation for $\boldsymbol{H}_Q^{-1}$ as

$$x_s\left[nT\right]=\boldsymbol{H}_Q^{-1}\left(x_d\left[nT\right]\right)\cong x_d\left[nT\right]=\hat{\boldsymbol{H}}_Q^{-1}\left(x_d\left[nT\right]\right) \quad . \tag{12}$$

Obviously, (12) has been obtained by neglecting the so-called quantization noise expressed by the component $e\left[n\right]$ in (8) or (9). Further, (12) can be viewed as an approximation of the inverse operator $\boldsymbol{H}_Q^{-1}$ by its pseudo-inverse counterpart $\hat{\boldsymbol{H}}_Q^{-1}$.

Let us now consider the inverse operators $\boldsymbol{H}_S^{-1}$. Its existence and form are nicely expressed by the sampling theorem [Nurfaizey et al. 2012] and the so-called cardinal series (the reconstruction formula [Nurfaizey et al. 2012]). As we know this series or formula is expressed in the following way [Nurfaizey et al. 2012]:

$$x\left(t\right)=\sum_{n=-\infty}^{\infty}x\left[nT\right]\frac{\sin\left(\frac{\pi}{T}\left(t-nT\right)\right)}{\frac{\pi}{T}\left(t-nT\right)} \quad . \tag{13}$$

Observe now that this formula can rewritten with the use of the operator terminology as

$$x(t)=\sum_{n=-\infty}^{\infty} x[nT]\frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)}= \\ =\sum_{n=-\infty}^{\infty} x_s[nT]\frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)}=\boldsymbol{H}_S^{-1}\left(x_s[nT]\right). \qquad (14)$$

Further, at the same time, the relation (14) constitutes a definition of the inverse operator $\boldsymbol{H}_S^{-1}$. We know from the sampling theorem [Nurfaizey et al. 2012] that this definition is correct in the sense that it enables a perfect reconstruction of a signal $x(t)$ when this signal is a lowpass one possessing a maximal frequency $f_{\max}$ in its spectrum that fulfills the following relation:

$$\frac{1}{T}=f_s\geq 2\cdot f_{\max}, \qquad (15)$$

where f_s denotes the sampling frequency.

Let us consider now forms of signals that are used in the network calculus. Generally, it can be said that a so-called cumulative traffic plays a role of a signal we know from the signal processing. And the former means a sum of packets (bits), which are entering or leaving a teletraffic system in a period from 0 to *t*, where 0 stands for an assumed initial moment. Further, it is clear that when *t* changes then this sum can be considered as a function of *t* and viewed as a kind of a teletraffic signal. Moreover, when this signal regards a system input it is then named an input traffic. In a similar way, a system's output traffic is defined.

For getting a cumulative teletraffic curve, we are counting bits or packets. Moreover, we know that any bit or packet has some length. So, let the bit or packet duration express in time units, say *T*, being just their lengths. This allows us to view the traffic as a function of time that has some granularity in both the value of its magnitude as well as in its argument *t*.

This fact is illustrated in Fig. 3 by a staircase curve form of a cumulative traffic $S(t)$.

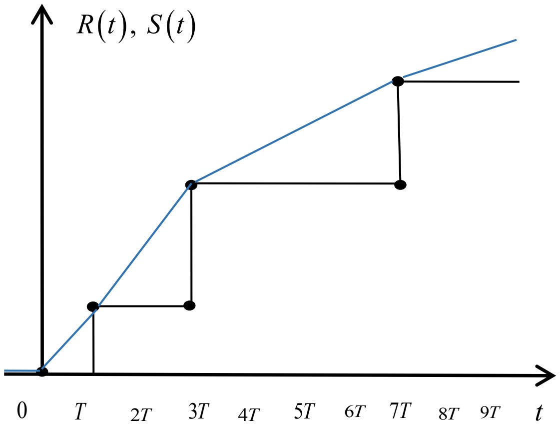

Fig. 3. Illustration to the natural form of a curve of the cumulative traffic in network calculus $S(t)$, indicated by a black line and its "fluid model" approximation $R(t)$, indicated by a blue line

The staircase form of the curve $S(t)$, shown in Fig. 3 is a natural one. However, in calculations performed in the network calculus, it is oft approximated by a continuous function of time $R(t)$ that is also shown in Fig. 3. The latter follows from applying the so-called fluid model in teletraffic modeling and analysis.

Observe from Fig. 3 that the function $R(t)$ is chosen in such a way that the following equality:

$$R(t = nT) = S(t = nT), \tag{16}$$

holds, where n means any integer. But, at all the remaining moments, the above functions can differ from each other. So, the problem of getting the function $R(t)$ from the values $S(nT)$ resembles the problem of recovery of the signal $x(t)$ from the samples $x_s[nT]$, which was discussed previously. It seems that one of the possible solutions to this problem by

$$R(t) = S(\lceil t/T \rceil T), \tag{17}$$

where $\lceil \cdot \rceil$ denotes the so-called ceiling function, is not satisfactory. The author of this paper is going to investigate this problem in more detail as well as a related one, which can be formulated as follows. Let $R'(t)$ be a system's output traffic obtained in calculations using this system description in terms of a fluid model by processing the input traffic $S(t)$. Further, let $R''(t)$ be an output traffic of the same system obtained now in another kind of calculations with the use of the staircase functions (natural ones for traffic systems). So, having this formulated, we can ask finally whether the following equalities

$$R'(t = nT) = R''(t = nT), \tag{18}$$

for any n, hold. This seems to be not obvious, it rather needs a proof.

3. SUMMARY

It has been shown in this paper that the operator theory can be successfully used in description of the interrelations existing between the basic operations performed on signals in the digital signal processing as well as in the network calculus. In this context, the two problems occurring in the network calculus, which are not yet satisfactory solved, have been indicated.

A short form of this paper indicates its preliminary character. In fact, that is the case. The work that is reported here will be continued. Investigations are planned, which will aim in exploration of more efficient digital signal processing and network calculus algorithms exploiting the operator formulations introduced in this paper. Preliminary findings are very promising.

Moreover, we hope to be able to solve some strictly theoretical problems occurring in the areas mentioned above with the help of the operator formalism introduced. The results of these endeavours will be published in the forthcoming papers.

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